Hydroelastic analysis of axially loaded Timoshenko beams with intermediate end fixities under hydrodynamic slamming loads

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\textbf{A B S T R A C T}

A theoretical hydroelastic analysis of an axially loaded uniform Timoshenko beam, with intermediate end fixities, undergoing hydrodynamic impact-induced bottom slamming, is presented. The underwater part of a marine craft is modeled as a lightly damped flexible beam, which rises out of the water in rough seas, and slams against it at a very large vertical velocity. This causes highly intense, localized hydrodynamic impact pressure sweeping across the beam at high velocities, setting it into high-frequency vibrations, predisposing it to plastic deformation and/or fatigue failure. The natural frequencies of the structure depend on the slenderness ratio, axial load, and end fixities. The natural frequencies and modeshapes are generated through the Eigen analysis. The changing wetted surface is the prime complexity of the problem. The relative velocity between fluid and structure is considered to establish the hydrodynamic pressure (radiation and impact). Normal mode summation method is used to analyze the transient structural vibration, for various impact speeds, deadrise angles, end fixities, and axial loads on the beam. The primary aim is to establish the zone of prominence of hydroelasticity, study the maximum dynamic stress magnitudes under various loading conditions and structural parameters; and draw conclusions leading to insights into sound structural designs.

\section{1. Introduction}

As conventional ship design has been giving way to non-conventional high performance marine vehicles, structural analysis of high speed crafts has become the cornerstone of a sound structural design. Planning crafts, hydrofoil crafts, catamarans, surface-effect ships (SES) are subject to various hydrodynamic loads. The dynamic lift due to planing leads to emergence of the craft above its zero-speed waterline and subsequent re-submergence. Transient slamming loads are much larger than wave loads, and act within a very short duration, predisposing the structure to local damage and fatigue loading. If the duration of the impact is close to the fundamental natural period of the structure, there is a pronounced hydroelastic interaction between the water and the structure. The sudden slamming load acts and decays, leaving the structure to vibrate at its natural frequencies. Axial tension in the structure makes it stiffer, moderating the fluid-structure interaction, and limiting the dynamic displacements and stresses. Axial compression makes the structure tender, making it susceptible to pronounced hydroelastic behavior, and generating large dynamic stresses. Since the impact-induced vibrations happen after the craft has penetrated the water surface, fluid inertia couples itself with the solid inertia, further reducing the fundamental natural frequency of the structure, leading to larger strains.

1.1. Timoshenko beam

Bokaian (1990) studied the free vibration of axially loaded Timoshenko beams, with classical end conditions. Lin (1994) studied the vibration of simply-supported Timoshenko beams to moving point loads, using Finite Element Analysis. This study was limited to a point load, and a single boundary condition of the beam, without any axial tension. Chang (1994) studied axially-loaded simply-supported Timoshenko beams on elastic foundations, but the external force was limited to point loads, varying randomly in time. Farchaly and Sgebl (1995) studied the frequencies and modeshapes of Timoshenko beams with intermediate fixities and elastic end supports. This study was limited to compressive axial load only. Wang (1997) studied the vibration of multi-span Timoshenko beams with intermediate fixities and elastic end supports. This study was comparatively done for both Euler-Bernoulli and Timoshenko beams but the spatial distribution of the stress evolution was not reported. Majkut (2009) solved the vibration of Timoshenko beams by the...
Green’s function method, since the forcing was a moving point load.

1.2. Slamming

Slamming is a non-linear phenomenon; the hydrodynamic pressure is non-linear with respect to the velocity potential associated with the fluid surrounding the structure. The radiation damping has both linear and non-linear terms w.r.t. the structural velocity, due to the hydroelastic coupling. However, the structural equations confirm to the linear equations of motion, since the response amplitudes are much smaller than the dimensions of the structure.

Rassinot and Mansour (1995) studied bottom slamming, using Strip theory for a Timoshenko beam, without axial load, with the classical boundary condition. Slamming has been extensively and intensively studied by Faltinsen (2002), Korobkin (2006), Khabakhpasheva and Korobkin (2003) and Khabakhpasheva (2013); including hydroelastic effects. The study has been done at the structural inertia phase, when the structure is still being wetted by the transient impact load. However, the above efforts focused on the fluid, and the structure was the basic Euler-Bernoulli beam, without any structural complications. Korobkin et al. (2006) dealt with the structural aspect of the problem by the numerical finite element method, without the Eigen expansion of the beam vibration and their influence on the added mass, radiation damping and hydroelastic loading. Faltinsen (1999) studied the hydroelastic response of orthotropic plates, using beam functions to generate the total hydrodynamic force, including hydroelastic effects. The study was also compared with experimental studies. Stenius et al. (2011) studied the hydroelastic behavior of panels to water impacts. Here, the kinematic and inertial hydroelastic effects were both considered distinctly. The variation of the maximum impact pressure over the structure, with and without hydroelastic effects, were compared for a single impact velocity and deadrise angle. Comparative dynamic loading factor plots for only one parameter, i.e. deadrise angle, were generated for the hydroelastic analysis and the rigid body analysis, and the range of prominence of hydroelasticity was highlighted.

1.3. Gap in literature and novelty of this work

The hydroelastic response and vibratory dynamics and their dependency on structural parameters; for an axially loaded Timoshenko beam, with non-classical (realistic) boundary conditions, under transient loads similar to the hydrodynamic impact configurations, has not been investigated as the authors know. Table 1 shows the comparison among various earlier works based on various aspects of the problem.

Table 1 also shows the highlights of this work as follows:

- The nature of the end fixities is crucial because the classical/ideal cases of simply supported or fixed nature are not possible in reality. The realistic boundary conditions are always intermediate and non-classical. They change the boundary stresses and strains. Therefore, an analysis of end fixities may lead to a more robust design.
- Since axial load influences the natural frequencies and the stress magnitudes, it should be included in the analysis.

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The impact at large vertical velocities, with small deadrise angles, causes high-order modes of vibration to be excited, where the shear deformation/rotary inertia are prominent. This necessitates the use of Timoshenko beam theory.

The hydrodynamic impact force has been modeled with the hydroelastic interaction for a Timoshenko beam. The wetting speed is moderated by the average structural velocity, which leads to five terms in the hydrodynamic force; rearranged as the added mass, damping, and impact force.

The generalized forces have been expressed with full mathematical rigor, involving Bessel and Struve functions w.r.t. the first wave number of the beam, and modified Bessel and modified Struve functions for the second wave number.

The response is studied over a wide range of impact duration or loading period of the beam. The dynamic loading factor is generated for various boundary conditions, axial loads, deadrise angles.

The zones of quasi-static behavior and large dynamic behavior have been demarcated. The zone of prominent hydroelasticity is important is highlighted. The kinematic and dynamic influence of the above three parameters is studied.

This work deals with ship bottom slamming, without forward speed effects. An impulse causes a structure to oscillate at all its natural frequencies. The duration of a hydrodynamic impact is of the order of milli-seconds, i.e. $T_{wp} \sim O(10^{-3}) \text{ s}$). Phenomena like air-trapping, water surface deformation, effects due to surface tension have been ignored in this analysis. Air-trapping is significant when the deadrise angle $\beta < 3^\circ$. This work is limited to 2D response to slamming, assuming the structure as a beam. The impact pressure is estimated from the Wagner's model, moderated by the relative velocity between the fluid and the structure. There are two time-scales in the slamming response phenomenon: (i) the duration of the sweep of the impact pressure (loading period) and (ii) the fundamental period of the structure. If the ratio of the impact duration to the first natural period is less than 2, hydroelasticity becomes important (Bereznitski, 2001).

The aim is to study the hydroelastic response under a transient impact load, and how it is affected due to realistic structural complications like (i) imperfect edge conditions, (ii) presence of axial loading, (ii) slenderness ratio of the beam. The output of the work focuses on the dynamic stress levels generated for the structure under various (a) boundary conditions, (b) axial loadings, (c) slenderness ratios, and (d) deadrise angles, (Figs. 6–9 respectively).

The steps followed in the work are as follows:

- **Free vibration analysis:** This is done by the Eigen expansion method, in order to evaluate the pairs of frequency parameters; generating the natural frequencies and modeshapes, with and without axial tension, for various rotational restraints at the ends.

- **Hydroelastic analysis:** The structure is then subjected to the hydroelastically-moderated Wagner’s configuration of the stretching hydrodynamic transient load at various deadrise angles (realistic case).

- **Dynamic Loading Factor (DLF):** The maximum dynamic deflection normalized by the maximum static deflection generates the DLF. The loading period has been non-dimensionalized by the first dry natural frequency of the beam, giving the non-D loading period $T$. The DLF vs. $T$ plots have been generated for several structural parameters.

2. Problem formulation

2.1. Structural modeling: axially loaded Timoshenko beam with non-classical boundary conditions

The underwater part of the vessel is assumed to be long enough such that slamming pressure is nearly 2D, and the response can be studied using Strip theory. The transverse section of the craft (Fig. 1) is modeled as a uniform, homogenous, prismatic Timoshenko beam, at a deadrise angle of $\beta$ to the horizontal; which impacts against the calm water surface, at a vertical velocity $V \text{ m/sec}$. The beam is axially loaded, either in compression or in tension, depending on the weight and buoyancy distributions. The length $L$ varies along $x(m)$, and the time is denoted as $t(\text{sec})$. The $x=0$ and $x=L$ locations typically correspond to the longitudinal stiffeners of the craft. Including the rise-up of the water due to the impact, the non-D loading period is: $T = \frac{2 \omega_{n1} L}{V_{s}}$, where $\omega_{n1}$ is the first dry natural frequency.

The boundary conditions vary between the two classical extremes: Simply-Supported (SS) beam to Clamped-Clamped (CC) beam, depending on the type of fixity (bracketing, sniping, etc.) and the quality of the weld. Study of beam with intermediate fixities, i.e. non-classical end conditions, is more realistic from the actual construction point of view. However, the other aspect of non-classical end conditions, i.e. elastically supported ends (Khabakhpasheva, 2013) or sliding ends (Stenius, 2011) is not studied here. The structure on which slamming occurs is a part of a hull, and does not move as a rigid body with respect to it. This study focuses on the local effects of the keel component of the hull.

Inclusion of axial load in the beam vibration analysis is practically relevant, since marine structural members are constantly under axial loads due to static pre-deflections, hogging/sagging of the keel plate, etc. Under the still water bending moment, the keel plate of the craft is in tension when the hull girder sags, and in compression when the hull girder hogs. This fact is further aggravated in the presence of wave bending moments. Several members of a marine craft may be pre-stressed due to rolling (during fabrication). The axial load (i) affects the natural frequencies of the beam, (ii) couples the beam modes in the normal mode summation method, and (iii) participates in the final bending stress distribution. The axial load is assumed to be uniform over the length of the beam, and over its cross-section. The magnitude of the tensile force considered in this work is equal to the critical buckling load $N_c$, which is known for SS and CC beams. The critical load magnitude for the intermediate end fixities are calculated through linear interpolation.

![Fig. 1. 2D slamming model of a typical high-speed craft (Hydroelastic interaction).](image-url)
2.2. Hydroelastic modeling

As seen in Engle and Lewis (2003), Wagner’s model over-predicts the impact load when the hydroelastic effects are significant. Hydroelastic interaction lowers the net impact pressure, reducing the structural strains as compared to those estimated by a dynamic analysis using 1-way coupling: the deformation reduces by 70% for \( \tau \approx 1 \), and about 20% for \( \tau \approx 2 \). It also lowers the natural frequency of the structure, and a tender structure suffers greater strains. In this work, we will be investigating the range of \( 0 < \tau < 2.5 \), where the role of hydroelasticity is important. The hydroelastic effects are particularly important for \( 0.5 < \tau < 1 \), where it amplifies the structural response.

The total slamming response consists of two parts: (i) the short-duration structural inertia phase, which can last only as long as the complete wetting of the structure by the moving load; and (ii) the longer free vibration phase, when the structural and hydrodynamic forces have steadied. In the first phase, the ‘rigid-body’ analysis over-predicts the total structural response; while in the second phase, it under-predicts the structural response. The structural inertia effects are important in the early stages of the impact. Thereafter the effects of structural velocity and fluid inertia increase in prominence; and finally, when the structure is fully wet, the fluid inertia dominates.

As explained in Stenius (2011), hydroelastic effects are of two types: (i) Kinematic: the structure deforms and changes the local impact velocity and the local deadrise angle, thereby changing the impact pressure envelope; (ii) Inertial: the ‘added’ mass stretches the time-scale w.r.t. the loading period, and causes the response to have a phase-lag with respect to the ‘rigid beam’ response. If the loading speed is more than the fundamental wet natural frequency of the structure, the impact force cannot be simply considered as a moving step function as shown in Jian and Grenestedt (2013), or a Dirac Delta function as shown in Rassinot and Mansour (1995). The hydrodynamic impact causes a high inertia increase in prominence; and for a Clamped-Clamped beam, the axial load is same in magnitude as the critical buckling load. For a Simply-Supported beam, the tensional axial force and negative for compressive axial force.

The forced vibration analysis is to be done for three different axial loads: zero, tensile, and compressive. The compressive load is 80% of the critical buckling load \( N_c \). For all intermediate fixities, the critical buckling load is linearly interpolated.

3. Analysis methodology

3.1. Free dry vibration

The governing differential equations (GDE) for free vibration of an axially loaded uniform Timoshenko beam, is:

\[
\rho A \frac{d^2 w(x, t)}{dt^2} = \mu AG \left( \frac{d^2 w(x, t)}{dx^2} - \frac{\partial \phi(x, t)}{\partial x} \right) + N \frac{d^2 w(x, t)}{dx^2},
\]

\[
\rho I \frac{d^2 \phi(x, t)}{dt^2} = \mu EI \left( \frac{d^2 \phi(x, t)}{dx^2} - \omega^2 \phi(x, t) \right) + EI \frac{d^2 \phi(x, t)}{dx^2}.
\]

(1A,B)

The material and geometric parameters and constants are: \( \rho \) = the density of the material, \( \rho \) = the cross-sectional area, \( I \) = the second moment of the cross-sectional area about the horizontal neutral axis, \( E \) = the elastic modulus, \( G \) = the shear modulus of the material, and \( \mu \) = the shear coefficient. The boundary conditions are non-classical with respect to rotation. The structure remains rigidly fixed to the hull. Thus, the displacement is zero at the ends, i.e.,

\[
w(0, t) = w(L, t) = 0;
\]

(2A,B)

and the bending moment equals the restraining moments, i.e.,

\[
EI \frac{d^2 w(0, t)}{dx^2} = k_{wl} \frac{\partial w(0, t)}{\partial x}; \quad EI \frac{d^2 w(L, t)}{dx^2} = -k_{wl} \frac{\partial w(L, t)}{\partial x}.
\]

(2C,D)

The left and right end fixities are modeled as torsional springs with torsional spring constants \( k_{wl} \) and \( k_{wr} \) respectively.

(i) As \( k_{wl} \) and \( k_{wr} \) \( \rightarrow 0 \), GDEs (1A,B) \( \rightarrow 0 \) and the beam approaches a Simply-Supported SS beam.

(ii) As \( k_{wl} \) and \( k_{wr} \) \( \rightarrow \infty \), GDEs (1A,B) \( \rightarrow \infty \) and the beam approaches a Clamped-Clamped CC beam.

This analysis is done with various end fixities. The fundamental wet natural frequency of the structure, the impact force cannot be simply considered as a moving step function as shown in Han et al. (1999).

3.2. Hydroelastic response

The hydrodynamic pressure configuration obeys the Wagner’s impact model (Faltinsen (1999)). The sweeping force rises to a peak value and drops to nearly zero. The (a) speed of loading (b) peak pressure, and (c) loading configuration; all three depend on the two
parameters (i) constant vertical impact velocity $V$ and (ii) deadrise angle $\beta$. The smaller the deadrise angle $\beta$, the greater is the peak pressure $P_{\text{peak}}$, more concentrated is the forcing configuration, and faster is the wetting. The jet head translation, along the ‘rigid’ beam, (Fig. 1) is given as

$$s(t) = \frac{V \pi}{2 \sin \beta}$$

Including the rise-up of the water due to the impact, the loading period and the non-dimensionalized loading period are:

$$T_{\text{loading}} = \frac{2 \sin \beta}{V \pi}, \quad t = \frac{2 \sin \beta}{V \pi} \frac{\omega}{2 \pi}$$

(7A,B)

Considering the Wagner’s pressure model, if we assume that the structural deflection does not affect the hydrodynamic.

pressure envelope, i.e. the impact pressure acts on a ‘rigid’ beam, then the following kinematic and dynamic formulations are valid. The peak and initial pressure are as follows:

$$P_{\text{peak}} = \frac{1}{2} \rho \left( \frac{dV}{dt} \right)^2, \quad P_{\text{initial}}(0, 0, t) = \rho V d V$$

(7C,D)

The pressure coefficient is expressed as $C_t = \frac{\rho \left( \frac{dV}{dt} \right)^2}{\pi^2} \left( \frac{\omega}{2 \pi} \right)^2$. But the Wagner’s model has been modified by the hydroelastic interaction, which leads to strong coupling among the modes of vibration, and non-linearity in the hydrodynamic forces w.r.t. the structural deformations. The total out-of-plane transverse flexural deflection of the beam is denoted as $\bar{w}(x, t)$, and the speed of this deformation is $\bar{w}(x, t)$. The impact velocity perpendicular to the beam is $v = \frac{\bar{w}(x, t)}{\cos \beta}$. As explained in Stenius et al. (2011), the relative velocity between the impact velocity and the mean rate of deformation of the structure is

$$\bar{w}(x, t) = \frac{V \pi}{2 \sin \beta} = \frac{V}{2} \frac{\omega}{2 \pi} \sin \beta$$

(8)

The average structural velocity is:

$$\bar{w}(x, t) = \frac{1}{s} \int_0^s \bar{w}(x, t) dx = \frac{1}{s} \int_0^s \sum_{j=1}^\infty \Phi_j(\gamma_j x) \bar{q}_j(t) dx = \frac{1}{s} \sum_{j=1}^\infty \frac{A_j}{\gamma_j} \cos(\gamma_j x) + \frac{B_j}{\gamma_j} \sin(\gamma_j x)$$

(9)

The average structural acceleration is:

$$\ddot{\bar{w}}(x, t) = \frac{1}{s} \int_0^s \ddot{\bar{w}}(x, t) dx = \frac{1}{s} \int_0^s \sum_{j=1}^\infty \Phi_j(\gamma_j x) \ddot{q}_j(t) dx = \frac{1}{s} \sum_{j=1}^\infty \frac{\Phi_j(\gamma_j x)}{\gamma_j} \ddot{q}_j(t)$$

(10)

The average structural velocity Eq. (9) and acceleration Eq. (10) are no longer functions of space, but of the wetted length ‘$s$‘. According to Wagner’s theory (Faltinsen, 1999), the velocity potential of the fluid on the wetted surface is:

$$\Psi_{\text{wet}}(x, t) = (\bar{w}(x, t)) \sqrt{(s(t)^2 - x^2)}$$

(11)

The linear hydrodynamic pressure on the wetted surface, by Bernoulli’s theory, is expressed as

$$p(x, t) = -\rho \left. \frac{\partial \Psi_{\text{wet}}(x, t)}{\partial t} \right|_t = -\rho \left( \frac{s(t)^2 - x^2}{\sqrt{(s(t)^2 - x^2)}} - (\bar{w}(x, t)) \frac{s(t)}{\sqrt{(s(t)^2 - x^2)}} \right)$$

(12)

There are three components of this pressure that generates the following three distinct forces:

**Virtual fluid inertia (added mass) component:** It is in phase with the structural acceleration $\ddot{w}(x, t)$, and acts as an “added mass” of a virtual added inertia i.e.

$$p_a = -\rho \bar{w}(x, t) \bar{w}(x, t) \sqrt{(s(t)^2 - x^2)}$$

(13)

**Radiation damping component:** The first term in the radiation damping is linear w.r.t. the average structural velocity $\bar{w}(x, t)$ while the second term in the radiation damping is non-linear w.r.t. the average structural velocity $\bar{w}(x, t)$

$$p_b = -\rho \bar{w}(x, t) \bar{w}(x, t) \sqrt{(s(t)^2 - x^2)} \frac{\omega}{2 \pi} \frac{\omega}{2 \pi}$$

(14)

**Hydrodynamic impact pressure component:** As seen below, there is a reduction/moderation in the impact pressure due to the velocity of the structure itself. This change in the impact pressure envelope is the kinematic effect of hydroelasticity (Stenius et al., 2011). As seen in Eq. (15), the second term in the impact force is identical to the linear damping term in Eq. (14). Hence, both are summed together as the radiation damping, and only the Wagner’s pressure (first term) is on the RHS:

$$F_I = \rho_a \bar{w}(x, t) \bar{w}(x, t) \sqrt{(s(t)^2 - x^2)} \frac{\omega}{2 \pi} \frac{\omega}{2 \pi}$$

(15)

The generalized pressures are calculated by pre-multiplying each of the pressure components (Eqs. (13), (14), (15)) with the beam modeshapes and integrating over the length L. The Generalized added mass, placed in the LHS of the GDE is

$$G\lambda_a = \rho \bar{w}(x, t) \sqrt{(s(t)^2 - x^2)} \frac{\omega}{2 \pi} \frac{\omega}{2 \pi} \left[ \frac{A_1 I_1(\gamma_1 x) + B_1 H_1(\gamma_1 x) + C_1 I_1(\gamma_2 x) + D_1 L_0(\gamma_2 x)}{\gamma_1 \gamma_2} \right]$$

(16)

Here, $I_1 = 1$st-order Bessel function of the 1st kind, $H_1 = 0$th-order modified Bessel function of the 1st kind, $L_0 = 0$th-order modified Struve function. The Generalized linear damping, placed in the LHS of the GDE, is

$$G\lambda_2 = -2 \rho \bar{w}(x, t) \bar{w}(x, t) \sqrt{(s(t)^2 - x^2)} \frac{\omega}{2 \pi} \frac{\omega}{2 \pi} \left[ \frac{A_2 I_0(\gamma_1 x) + B_2 H_0(\gamma_1 x) + C_2 I_0(\gamma_2 x) + D_2 L_0(\gamma_2 x)}{\gamma_1 \gamma_2} \right]$$

(17)

The Generalized impact pressure, placed on the RHS of the GDE, is

$$G\lambda_3 = \rho \bar{w}(x, t) \bar{w}(x, t) \sqrt{(s(t)^2 - x^2)} \frac{\omega}{2 \pi} \frac{\omega}{2 \pi} \left[ \frac{A_3 I_1(\gamma_1 x) + B_3 H_1(\gamma_1 x) + C_3 I_1(\gamma_2 x) + D_3 L_0(\gamma_2 x)}{\gamma_1 \gamma_2} \right]$$

(18)
The beam modeshapes and their respective curvatures are not orthogonal to each other. We know \( \int_k \Phi \Phi' dx = 0 \) if \( k \neq j \). But, \( \int_k \Phi \Phi'' dx \neq 0 \) even when \( k = j \). Writing Eq. (22A,B) in the matrix form, the modal GDE is:

\[
\begin{align*}
\sum_{j=1}^{\infty} \mu_{AG} \Phi_j \Phi_j dx; & \quad \sum_{j=1}^{\infty} \gamma_j \gamma_j dx; \\
\sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; & \quad \sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; \\
\sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; & \quad \sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; \\
\sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; & \quad \sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx;
\end{align*}
\]

The coupled system of governing differential equation (GDE) of impact-induced undamped forced vibration of an axially loaded Timoshenko beam, utilizing Newton’s second law, is given by:

\[
\rho_d \frac{d^2 \Phi(x,t)}{dt^2} = \mu_{AG} \left[ \frac{d^2 \Phi(x,t)}{dx^2} - \rho \frac{d \Phi(x,t)}{dx} \right] + N \frac{d^2 \Phi(x,t)}{dx^2} + F(x,t) + F_B(x,t), \quad (20A)
\]

\[
\rho_l \frac{d^2 \gamma(x,t)}{dt^2} = \mu_{AG} \left[ \frac{d \gamma(x,t)}{dx} - \rho \frac{\gamma(x,t)}{dx} \right] + E_l \frac{d^2 \gamma(x,t)}{dx^2} \quad (20B)
\]

The equations are found in Farchalny and Segl (1995), Majkut (2009). Using Eq.3(A,B) in Eq. (20), the GDE-pair becomes:

\[
\sum_{j=1}^{\infty} \mu_{AG} \Phi_j \Phi_j dx; + \sum_{j=1}^{\infty} \Phi_j \Phi_j dx; + \sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; + \sum_{j=1}^{\infty} \gamma_j \gamma_j dx; = 0
\]

The dynamic normal stress is expressed as:

\[
\sigma(x, \tau) = \frac{N}{A} + \frac{E}{A} \sum_{j=1}^{\infty} \frac{d^2 \Phi_j(x)}{dx^2} - \frac{\Phi_j(x)}{\rho \frac{d \Phi_j(x)}{dx}}
\]

The dynamic shear stress is expressed as:

\[
\tau(x, \tau) = \mu G \left[ \frac{d \gamma(x,t)}{dx} - \frac{\gamma(x,t)}{\rho \frac{\gamma(x,t)}{dx}} \right] + E_l \frac{d^2 \gamma(x,t)}{dx^2}
\]

3.3. Static analysis (Shear Beam)

The shear deformation \( \mu_{AG} \frac{d^2 \gamma(x,t)}{dx^2} \) gets included here. However, the rotary inertia \( \frac{d^2 \gamma(x,t)}{dx^2} \), being a dynamic term, does not appear. The system of equations of forced deflection of an axially loaded Timoshenko beam is:

\[
0 = \rho G \left[ \frac{d^2 \gamma(x,t)}{dx^2} - \frac{\gamma(x,t)}{\rho \frac{\gamma(x,t)}{dx}} \right] + E_l \frac{d^2 \gamma(x,t)}{dx^2}
\]

The beam modeshapes \( \Phi_j(x) \) and slope modes \( \Theta_j(x) \) to generate the deflection and bending slope respectively.

However, the generalized axial load GN is non-diagonal, since the beam modeshapes and their respective curvatures are not orthogonal to each other. We know \( \int_k \Phi \Phi'' dx = 0 \) if \( k \neq j \). But, \( \int_k \Phi \Phi'' dx \neq 0 \) even when \( k = j \). Writing Eq. (22A,B) in the matrix form, the modal GDE is:

\[
[GM_j\Phi_j(x) + GA_j \Phi_j(x) + \sum_{j=1}^{\infty} \Phi_j \Phi_j dx; + \sum_{j=1}^{\infty} \mu_{AG} \gamma_j \gamma_j dx; = 0
\]

The beam modeshapes \( \Phi_j(x) \) are orthogonal to each other, which renders the generalized mass GM, generalized stiffness GK diagonal.
The maximum dynamic deflection is normalized by the maximum static deflection to generate the Dynamic Load Factor (DLF). The maximum dynamic normal stress (Eq. 24) has been normalized by the maximum static normal stress (Eq. 27(A)). Thus,

$$DLF = \max \left[ \frac{w(x, t)}{\max \{w_0(x, t)\}} \right]$$

The tensile stress DLF is

$$DLF = \max \left[ \frac{\sigma(x, t)}{\max \{\sigma_0(x, t)\}} \right]$$

Shear Stress DLF is

$$DLF = \max \left[ \frac{\tau(x, t)}{\max \{\tau_0(x, t)\}} \right]$$

4. Results

The geometric and material properties of the Timoshenko beam are: $h = 1$ m, square cross-section $0.02 \times 0.02$ m, $\rho = 7850$ $kg/m^3$, $E = 209$ GPa, Poisson’s ratio $\nu = 0.3$. The parametric space is: (i) $0 < N_c < 2$; (ii) $N = N_c, N_c - 0.8N_c$; (iii) $h = h = 0.05$. The impact pressure configuration is based on the following parameters: $\beta = 5^\circ, 10^\circ, 15^\circ$, and $0 < \tau < 5$. The first two beam modes have been used, as justified later.

4.1. Timoshenko beam

Table 2 shows the first two dry natural frequencies of the Timoshenko beam with various non-classical end conditions, i.e. $0 < \log_{10}K_p < 2$. The maximum dynamic deflection is normalized by the maximum static deflection to generate the Dynamic Load Factor (DLF). The maximum dynamic normal stress (Eq. 24) has been normalized by the maximum static normal stress (Eq. 27(A)). Thus,

$$DLF = \max \left[ \frac{w(x, t)}{\max \{w_0(x, t)\}} \right]$$

The tensile stress DLF is

$$DLF = \max \left[ \frac{\sigma(x, t)}{\max \{\sigma_0(x, t)\}} \right]$$

Shear Stress DLF is

$$DLF = \max \left[ \frac{\tau(x, t)}{\max \{\tau_0(x, t)\}} \right]$$

Table 3 shows the kinematic and dynamic characteristics of the hydrodynamic slamming. The Timoshenko beam is clamped-clamped at the edge and has a slenderness of 0.02. The deadrise angle $\beta = 10^\circ$. Three different non-D loading periods have been chosen for our study, i.e. $\tau = 1, 1.8, 2.5$. The largest dynamic overshoot over the static deflection is expected at $\tau = 1$. For $\tau = 2.5$, the peak pressure is lowered and is further moderated by the hydroelastic interaction, a quasi-static behavior is expected. The smaller the $\tau$, the less is the hydroelastic influence, since the impact sweep is too fast for structure to react and alter the impact pressure envelope. With axial tension, a larger loading period $T_p$ leads to the same $\tau$; causing a greater impact velocity and pressures. The opposite is true with axial compression.

Fig. 4(a)–(c) show the impact pressure configuration, with and without hydroelastic interaction, on a CC beam, at $\beta = 10^\circ$, and $\tau = 1, 1.8, 2.5$. The “rigid-body” Wagner’s peak and initial pressure are: $P_{peak} = 79.36$, $P_{max} = 17.82$. The hydroelastic interaction causes a moderation in the peak pressure as well as the initial pressure. The impact pressure envelope is modified due to the flexural response of the structure. The slower the loading, the more is the modification. The peak impact pressure on a “rigid-body”, as calculated by the Wagner’s method, matches well with those reported experimentally by Faltinsen (1999) and numerically by Stenius et al. (2011). According to Faltinsen (1999), for $V = 2.6$ m/s, $tan\beta = 0.25$, $P_{peak} = 135$ KPa. Here, $V = 2.6$ m/s, $tan\beta = 0.25$, $P_{peak} = 136.77$ KPa. According to Stenius et al. (2011), for $V = 6$ m/s, $\beta = 30^\circ$, $P_{peak} = 127.30$ KPa. Here, $V = 6$ m/s, $\beta = 30^\circ$, $P_{peak} = 136.57$ KPa. All the three cases here have a faster loading than that

- **Kinematic**: changes in frequency shift the non-D loading period, i.e. stretches/compresses the time scale.
- **Dynamic**: A softer beam suffers a greater dynamic deflection than a stiffer beam, especially in the zone $0.5 < \tau < 2$.

Fig. 2 shows the non-dimensional frequency parameter, calculated from the two wave numbers of the beam $f_l$ and $f_p$. The slenderness ratio is $L = 0.02$, which is well within the range of ‘pure bending’ assumption holding true. The fixity ratio is the non-dimensional $K_p = \frac{w_{max}}{w_{0}}$. As $K_p$ is reduced, the beam begins to behave like a simply-supported (SS) beam, with its fundamental frequency parameter approaching the classical value of $\pi/2$. As $K_p$ is increased, it begins to behave like clamped-clamped (CC) beam, with its fundamental frequency parameter approaching the classical value of $4.73$. The same trend is seen for the second frequency parameter, which approaches the value of $2\pi$ for an SS beam and the classical value of 7.85 for a CC beam. The ‘transition’ zone is demarcated as $0 < \log_{10}K_p < 2$, for both the modes.

Fig. 3 shows the first two modeshapes of a Timoshenko beam, for various end fixities: $\log_{10}K_p = 0, 1, 2$. With the ends hinged, the modeshapes follow the typical sinusoidal nature of a SS-beam. As the end fixity increases, the ends develop curvature and the end slopes decrease to zero, leading to a typical CC-beam modeshape. It is seen that for $\log_{10}K_p = 1$, the beam modeshape characteristics is in between those of the two classical beam modeshapes.

4.2. Hydroelastic loading and response

Table 3 shows the kinematic and dynamic characteristics of the hydrodynamic slamming. The Timoshenko beam is clamped-clamped at the edge and has a slenderness of 0.02. The deadrise angle $\beta = 10^\circ$. Three different non-D loading periods have been chosen for our study, i.e. $\tau = 1, 1.8, 2.5$. The largest dynamic overshoot over the static deflection is expected at $\tau = 1$. For $\tau = 2.5$, the peak pressure is lowered and is further moderated by the hydroelastic interaction, a quasi-static behavior is expected. The smaller the $\tau$, the less is the hydroelastic influence, since the impact sweep is too fast for structure to react and alter the impact pressure envelope. With axial tension, a larger loading period $T_p$ leads to the same $\tau$; causing a greater impact velocity and pressures. The opposite is true with axial compression.

Fig. 4(a)–(c) show the impact pressure configuration, with and without hydroelastic interaction, on a CC beam, at $\beta = 10^\circ$, and $\tau = 1, 1.8, 2.5$. The “rigid-body” Wagner’s peak and initial pressure are: $P_{peak} = 79.36$, $P_{max} = 17.82$. The hydroelastic interaction causes a moderation in the peak pressure as well as the initial pressure. The impact pressure envelope is modified due to the flexural response of the structure. The slower the loading, the more is the modification. The peak impact pressure on a “rigid-body”, as calculated by the Wagner’s method, matches well with those reported experimentally by Faltinsen (1999) and numerically by Stenius et al. (2011). According to Faltinsen (1999), for $V = 2.6$ m/s, $tan\beta = 0.25$, $P_{peak} = 135$ KPa. Here, $V = 2.6$ m/s, $tan\beta = 0.25$, $P_{peak} = 136.77$ KPa. According to Stenius et al. (2011), for $V = 6$ m/s, $\beta = 30^\circ$, $P_{peak} = 127.30$ KPa. Here, $V = 6$ m/s, $\beta = 30^\circ$, $P_{peak} = 136.57$ KPa. All the three cases here have a faster loading than that
from Stenius et al. (2011), in which \( V = 6 \text{ m/sec}, \) and \( \beta = 30^\circ \). Given their structural characteristics, the \( \tau = 3.68, \) leading to the experimental \( \tau = 7.04, \) which was verified with our model. A longer loading duration allows an appreciable hydroelastic interaction, seen as the modification of the impact pressure envelope and increase in the initial pressure (Fig. 4). A fast sweep of the impact load keeps the initial pressure nearly unchanged. A slower sweep enhances the initial pressure, as seen in Fig. 4(c) and seen experimentally in Stenius et al. (2011). For all loading periods, the fundamental mode generates a larger generalized impact pressure as compared to the second mode. The loading configuration has a strong spatial alignment with the first modeshape. The second modeshape, with a node at \( x/L = 1/2 \), aligns moderately with the impact force. The higher modes, with more nodes, align far less than the first two modes, and have hardly any contribution to the total deflection. Thus they are ignored in the analysis.

Fig. 5(a) shows the time-evolution of the dynamic deflection \( w(L/2, t) \) during the wetting of the beam \( \tau = 1.0, 1.8, 2.5, \) through both 1-way coupling (dynamic analysis) and 2-way-coupling (hydroelastic analysis), for no axial loading. It is seen that the hydroelastic time-evolution of the deflection is delayed with respect to the ‘rigid-body’ time-evolution. The fluid appreciably reduces the natural frequency of the structure, making it softer than the ‘dry’ beam. The added-mass component stretches the time-scale of the response. At \( \tau = 1.0 \), it seen that w.r.t to the maximum hydroelastic deflection, the maximum dynamic deflection is higher by about 20%. At \( \tau = 1.8, \) the influence of hydroelasticity is seen to reduce from the dynamic point of view. In the quasi-static range, i.e. \( \tau = 2.5, \) both analyses predict nearly the same maximum deflection magnitude. Here, the fluid radiation forces (added

Fig. 4. (a,b,c) Impact pressure configuration with hydroelastic interaction (solid), and without hydroelastic interaction (dashed), at non-D loading period \( \tau = 1.0 \) (left), 1.8 (right), 2.5 (below); deadrise angle \( \beta = 10^\circ \).

Fig. 5. (a) Dynamic deflection \( w(L/2, t) \) for \( \tau = 1.0, 1.8, 2.5, N=0 \), (b) Dynamic deflection \( w(L/2, t) \) for \( \tau = 1.0, N=N_c, 0, -0.8N_c \), (c) Dynamic deflection \( w(L/2, t) \) for \( \tau = 2.5, N=N_c, 0, -0.8N_c \).
mass and damping) are negligible, and the hydroelasticity effects can be ignored. For $\tau > 3$, the beam vibratory response is quasi-static. Here, the effect of hydroelasticity is minimal, with the deflection $w(L/2)$ evolving almost similarly from both dynamic and hydroelastic analyses, without much phase difference. Interestingly, for $\tau < 0.5$, the response is less than that predicted by an equivalent static analysis. The hydroelastic effect is more pronounced and significant in the range $0.5 < \tau < 2$ (Berezinski, 2001). Here, this will be termed as the ‘range of interest’. In most of our range of interest, where the hydroelasticity is prominent, the 1-way dynamic analysis of a ‘rigid’ beam over-predicts the maximum stress levels. Both normal stress and shear stress time-evolutions follow the pattern of the deflection, with the hydroelastic stress developing with a phase-lag with respect to the dynamic stress. The normal stress trends have been verified with Korobkin (2003).

Fig. 5(b) and (c) show the kinematic and dynamic influences of axial loading on the hydroelastic response of the Clamped-Clamped Timoshenko beam for two different non-D loading periods, i.e. $\tau = 1.0$ where there is a large dynamic overshoot of the response, and $\tau = 2.5$, where the response is nearly quasi-static. For three different axial loadings $N = N_\alpha, 0.8N_\alpha$, the three pairs of deflection-time-evolutions, for both 1-way and 2-way coupling, have been shown. Axial compression gives a softer beam, leading to a longer wetting time for the same load. It also allows larger deflections, both with and without hydroelastic interaction. Axial tension gives a stiffer beam, leading to a shorter wetting time for the same load. At $\tau = 1.0$, the hydroelastic influence is far more pronounced than at $\tau = 2.5$, irrespective of the axial loading.

4.3. Dynamic loading factor

Fig. 6(a) shows the Dynamic loading factor of a Timoshenko beam, without axial load, for a deadrise angle of $\beta = 10^\circ$, for various non-classical edge conditions of the beam, $log_{10}K_\beta=6,0,1.2,6$. The DLF for $log_{10}K_\beta=-6,1,2,6,\beta=10^\circ, N=0$. are coincident, depicting the typical hydroelastic response spectrum of a Simply-Supported Timoshenko beam to a hydrodynamic impact-induced slamming load. The DLF for $log_{10}K_\beta=2,6$ are coincident, depicting the typical hydroelastic response spectrum of a Clamped-Clamped Timoshenko beam to the same load. For $log_{10}K_\beta=1$, the DLF characteristic trends midway between the classical beam DLF trends, in our range of interest, i.e. $0.5 < \tau < 2$. In this zone, a stiffer beam is seen to have a higher deflection and thus, higher stress levels. Since the X-axis is the loading period non-dimensionalized by the natural period of the beam, a stiffer beam stretches the X-axis towards the right. As seen in the authors’ previous work, Datta and Siddiqui (2013), a dynamic analysis lead to a maximum DLF of ~1.8 in a “rigid” CC beam, and ~1.7 in a “rigid” SS beam. But a hydroelastic analysis stems the response, showing a maximum DLF of ~1.25 in a CC beam, and ~1.2 in an SS beam. Faltsinsen (1999) established a DLF with ~22% overshoot above the quasi-static analysis (DLF ~ 1.22) for an orthotropic plate, using CC Timoshenko beam functions, verified with experiments. For $\tau < 1$, i.e. the impact velocity increases, the DLFs steeply descend to zero for $\tau=0$. This trend is again similar to Faltsinsen (1999). The impact sweeps the beam so quickly, that the structure is unable to react appreciably, while the corresponding static analysis over-predicts the deflection under a huge impact pressure, which is proportional to the square of the impact velocity. In this range, a softer beam would produce a slightly higher DLF.

Fig. 6(b) and (c) show the corresponding DLF for the maximum tensile stress and the maximum shear stress, respectively, of a Timoshenko beam, for $log_{10}K_\beta=6,0,1,2,6$. The trends are similar as in Fig. 6(a). The stress DLFs are seen to closely follow the dynamic deflection DLFs, and hence are presented for the axially unloaded beam only.

Fig. 7(a)–(c) shows the DLFs with various end fixities, i.e. $log_{10}K_\beta=6,1,6$, for a deadrise angle of $\beta = 10^\circ$, each for various axial
loadings on the beam, i.e. \( N = N_c, 0, -0.8N_c \). Axial loading on a SS-beam (Fig. 7(a)) has a mild influence on the DLF characteristics. The axially compressed beam suffers a larger DLF at \( \tau \leq 1 \), while the axially tensed beam has a very slightly higher DLF in the range \( 1 < \tau < 2.5 \). On the other extreme, axial loading on a CC-beam (Fig. 7(c)) causes a more prominent influence on the \( \tau \)-axis and the three DLF characteristics become distinct. The peak DLF (for any \( K_{\theta} \)) increases marginally for decreasing axial load. Increasing the end fixity also increases the \( N_{cr} \) of the beam, indicating the plots in Fig. 7(a) have a smaller difference of axial load among them, as compared to the plots in Fig. 7(c).

4.3.1. Kinematic effect

Axial tension stretches the non-D loading period \( \tau \)-axis against the DLF, and axial compression shrinks it. The peak DLF (for \( N < 0 \)) increases marginally for increasing end fixities and shifts slightly to a lower \( \tau \). But, the peak DLF (for \( N > 0 \)) increases marginally for increasing end fixities and shifts slightly to a higher \( \tau \). For the same \( \tau \), axial tension gives a smaller loading period \( T_{sp} \) and hence a greater impact sweep velocity and hence greater slamming pressures. But for the same \( \tau \), axial compression gives a larger loading period \( T_{sp} \) and hence a smaller impact sweep velocity and hence lower slamming pressures (Eq. (19)). The non-D loading period uses the dry natural period of the beam, and thus, the \( \tau \)-axis is independent of the added mass effect. In the zone \( 1.2 < \tau < 2.5 \), axial tension leads to a consistently higher DLF, due to the stretching of the DLF characteristics with respect to the \( \tau \)-axis.

4.3.2. Dynamic effect

As the end fixity increases, the critical buckling load also increases. A softer beam will deflect more and hence will develop more stresses than a stiffer beam. It will attain a higher DLF at a lower \( \tau \). For the range \( 0.5 < \tau < 1.2 \), axial compression causes a larger dynamic loading factor, since the beam is softer and suffers higher deflections. But also, for the same \( \tau \), axial tension gives a lower DLF since the stiffer beam resists the impact. Beyond \( \tau \approx 3 \), the response is quasi-static, irrespective of the end fixity or axial loading.

Fig. 8 shows the DLF of a CC Timoshenko beam, without axial load, for a deadrise angle of \( \beta = 10^\circ \), for various slenderness ratios of the beam, i.e. \( \beta=0.05,0.1,0.2 \). A stockier beam is stiffer, leading to the stretching of the DLF characteristic over the non-D loading period \( \tau \). At \( \tau > 5 \), the response is quasi-static irrespective of the beam thickness. Similarly for \( \tau < 1 \), the DLF characteristics coincide for all slenderness

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Fig. 8. Hydroelastic DLF of Clamped-Clamped Timoshenko beam; \( \beta = 10^\circ \), \( h/L = 0.05,0.1,0.2 \), no axial load.

Fig. 9. Hydroelastic DLF of Clamped-Clamped Timoshenko beam; \( \beta = 5^\circ, 10^\circ, 15^\circ \); \( \beta=0.05 \), no axial load.

Fig. 10. Deflection \( d(t) \) (normalized by max static deflection) for sweep length \( s/L = 0.25, 0.5, 0.75, 1.0 \); \( \beta=9^\circ \), \( V = \omega n L / \pi \).

Fig. 11. Bending Moment \( M(t) \) (normalized by max static BM); for sweep length \( s/L = 0.25, 0.5, 0.75, 1.0 \); \( \beta=9^\circ \), \( V = \omega n L / \pi \).
The non-D loading period $\tau$; $V$ have been demarcated. They are (i) the static over-prediction (very safe) zone, (ii) the hydroelastic zone, and (iii) quasi-static (safe) zone.

4. Axial loading has been included, which changes the natural frequency, and hence the non-dimensional loading period.

The designer aims to design structure with the boundary conditions and, stiffness and damping such that the composite time-scale $\tau$ is greater than 2.5 for the most probable impact velocities. To operate in the quasi-static range, the forcing speed should be low, or the natural frequency of the beam must be high. A craft meant for inland operations (i.e. calm waters) can afford a softer structure, where slamming is less common. A sea-going vessel, on the other hand would require a stiffer bottom structure to ensure a quasi-static response. High-speed vessels, though use mostly in calm waters, needs stiffer material, since the varying dynamic lifts (partially supporting the craft weight) cause repeated slamming of the bow at high impact speeds.

References