Free transverse vibration of ocean tower

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ABSTRACT

This paper studies a continuous, elastic model of an ocean tower, partially submerged in water, undergoing free transverse vibration in a plane. The tower is modeled as a non-uniform Timoshenko beam, which is supported by an eccentric tip mass on one end and a non-classical damped foundation on the other. The tower is modeled as a combination of translational and rotational springs and dampers. The effect of shear deformation and rotary inertia is included in the analysis. The free vibration equation is derived using Hamilton’s variational principle based on two approaches, Rayleigh Ritz Method (RRM) and Finite Element Method (FEM), which show a good agreement in results. The computational efficiency of RRM over FEM is shown using a convergence study. Finally, a parametric study is done to demonstrate the dependence of natural frequency on different configurations of the tower.

1. Introduction

The dynamic behavior of structures like ocean tower is an area of extensive research. These structures are widely used to support superstructures like wind turbine, bridges, etc. The ocean towers are often subjected to dynamic loads such as wind, waves, etc. Hence, it is important to analyze the dynamic behavior of such structures. These dynamic behaviors can be predicted with reasonable accuracy if the structures are modeled as beam with tip mass and non-classical foundation. Hence, a lot of research has been done to study the problem in this field. For example, Wu and Hsu (2007) analyzed the free vibration of partially wet, elastically supported uniform Euler–Bernoulli beam with eccentric tip mass using two separate sets of analytical formulation. Uscilowska and Kołodziej (1998) provided closed form solution for a partially immersed cantilever beam with eccentric tip mass. Auciello and Ercolano (2004) provided solution for the non-uniform Timoshenko beam to solve the free vibration by the energy method. Wu and Chen (2005) solved the free vibration of non-uniform partially wet Euler–Bernoulli beam with elastic foundation and tip mass. De Rosa et al. (2013) calculated closed form solution for free vibration of a linearly tapered, partially immersed, elastically supported column (Euler–Bernoulli beam) with eccentric tip mass. Wu and Chen (2010) studied the wave-induced vibrations of an axially loaded, immersed, uniform Timoshenko beam carrying an eccentric tip mass with rotary inertia using analytical formulation.

The vibration analysis of non-uniform Timoshenko beam as ocean tower has been rarely investigated as the authors know. In this work, the ocean tower is modeled as a partially submerged, non-uniform Timoshenko beam supported by a rigid tip mass with eccentricity at the free end, and non-classical damped foundation on the other end. The effect of shear deformation and rotary inertia is included in the beam. The free vibration equation is derived using Hamilton’s variational principle based on Rayleigh Ritz Method (RRM) and Finite Element Method (FEM). The solution obtained by these two approaches show a good agreement in results. In RRM, the trial function, which is used to obtain non-uniform beam mode-shapes of ocean tower, is assumed as uniform beam mode-shapes satisfying the boundary conditions of ocean tower. In FEM, the Mindlin-type linear beam element of C0-order with four degrees of freedom, as explained in Bathe (1996), has been used as shape function. In order to avoid shear locking, reduced integration technique has been incorporated in FEM as explained later. Some of the results are also compared with one present in the existing literature for the verification of computer program and model.

The methodology (Hamilton’s variational principle) involves an integral equation and hence, higher order non-uniformity in section area of beam can be handled easily. In both the approaches, i.e., RRM and FEM, the free vibration equation is derived by using this methodology. The difference between RRM and FEM lies in choosing the shape function for finding the solution. In RRM, the shape function is chosen over the entire beam while in FEM, it is chosen only for one element. Hence, the main challenge of finding the solution through RRM is satisfying all the boundary conditions using the same shape function. However, the benefit lies in its computational efficiency which is much higher than FEM. This is shown in the convergence study (Section 4.2) where the solution obtained by RRM...
2. Problem formulation

A continuous, elastic model of an ocean tower is shown in Fig. 1.
The tower is modeled as Timoshenko beam. The length of the tower
without the tip mass is \( L \). It is immersed up to \( \alpha L \). The foundation of the
tower is partially constrained against translation and rotation. Hence, it
can be modeled as a combination of translational and rotational linear-
springs, with spring constants \( k_t \) and \( k_r \), respectively. To account for
damping effect due to loose silt at sea bed, translational and rotational
linear-dampers, with damping constants \( c_t \) and \( c_r \), respectively are

![Image](329x61 to 509x218)

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**Nomenclature**

- **\( x \)**: space variable along the beam length.
- **\( t \)**: time variable.
- **\( u \)**: transverse deflection of non-uniform beam.
- **\( \theta \)**: pure bending slope of non-uniform beam.
- **\( q \)**: principle coordinate of non-uniform beam.
- **\( \omega \)**: natural frequency of non-uniform beam.
- **\( \omega_R \)**: real part of \( \omega \).
- **\( \omega_I \)**: imaginary part of \( \omega \).
- **\( L \)**: length of the beam without tip-mass.
- **\( D_b \)**: diameter of the base of the tower.
- **\( D_p \)**: diameter of the upper uniform part of the tower.
- **\( I_p \)**: reference tip mass.
- **\( I_p^0 \)**: reference rotary inertia of tip mass.
- **\( m_p \)**: tip mass.
- **\( I_p \)**: rotary inertia of tip mass.
- **\( \beta \)**: tapering ratio.
- **\( \gamma \)**: tip mass ratio.
- **\( \zeta \)**: rotary inertia ratio of tip mass.
- **\( \rho \)**: density of steel.
- **\( \rho_w \)**: density of water.
- **\( M \)**: bending moment in non-uniform beam.
- **\( A \)**: section area of non-uniform beam.
- **\( i \)**: sectional area 2nd moment of non-uniform beam.
- **\( C_A \)**: added mass coefficient.
- **\( k_{t0} \)**: reference translational spring constant.
- **\( k_{r0} \)**: reference rotational spring constant.
- **\( k_t \)**: translational spring constant of non-uniform beam.
- **\( k_r \)**: rotational spring constant of non-uniform beam.
- **\( \kappa \)**: Foundation spring stiffness ratio.
- **\( \kappa' \)**: foundation spring stiffness ratio as used by Wu and Chen (2010).
- **\( c_t \)**: translational damping constant of non-uniform beam.
- **\( c_r \)**: rotational damping constant of non-uniform beam.
- **\( \eta_r \)**: proportionality constant of \( c_r \).
- **\( \eta_t \)**: proportionality constant of \( c_t \).
- **\( E \)**: modulus of elasticity of the material.
- **\( G \)**: shear modulus.
- **\( k_0 \)**: shape factor of the cross-section.
- **\( \nu \)**: Poisson’s ratio.
- **\( u_f \)**: transverse deflection of uniform beam.
- **\( \theta_f \)**: pure bending slope of uniform beam.
- **\( q_f \)**: principle coordinate of uniform beam.
- **\( \omega_f \)**: natural frequency of uniform beam.
- **\( \omega_{fR} \)**: real part of \( \omega_f \).
- **\( \omega_{fI} \)**: imaginary part of \( \omega_f \).
- **\( M_f \)**: bending moment in uniform beam.
- **\( V_f \)**: shear force in uniform beam.
- **\( A_f \)**: section area of uniform beam.
- **\( I_f \)**: sectional area 2nd moment of uniform beam.
- **\( g \)**: gravitational acceleration.
- **\( \varphi \)**: uniform beam mode-shape (trial function).
- **\( \phi \)**: non-uniform modeshape.
- **\( \psi \)**: non-uniform pure bending slope mode-shape (trial function).
- **\( n \)**: no. of trial functions considered.
- **\( U \)**: potential energy.
- **\( T \)**: kinetic energy.
- **\( R \)**: Rayleigh dissipation factor.
- **\( M \)**: mass matrix.
- **\( C \)**: damping matrix.
- **\( K \)**: stiffness matrix.
- **\( \xi \)**: local space coordinate.
- **\( \xi' \)**: local space coordinate.
- **\( \xi'' \)**: local space coordinate.
- **\( \xi''' \)**: local space coordinate.
- **\( L_e \)**: length of the beam element.
- **\( U_e \)**: potential energy of beam element.
- **\( T_e \)**: kinetic energy of beam element.
- **\( R_e \)**: Rayleigh dissipation factor of beam element.
- **\( W_g \)**: work done due to gravity on beam element.
- **\( n_e \)**: Nth element of the beam.
- **\( N_e \)**: total number of beam elements.
included. The magnitude of the spring and damping constants can be determined experimentally. These constraints influence the natural frequencies and mode-shapes of the structure. The inclusion of the rigid eccentric tip mass, of mass \( m_p \), rotary inertia \( I_p \) and eccentricity \( e_p \) decreases the natural frequency since it adds to the kinetic energy without changing the potential energy of the structure. For simplicity, the section diameter increases linearly from \( x=\beta L \) to \( x=0 \) so as to decrease the stress level at the foundation, though it will be seen that the methodology can handle non-linear variation in section diameter as well. For the underwater part of the beam, effective density of the added mass coefficient \( C_A \) of the model can be determined experimentally. In the present work, \( C_A = 1 \) as given by Uscilowska and Kolodziej (1998).

3. Analysis methodology

3.1. Rayleigh Ritz Method (RRM)

The displacement \( u \) and the pure bending slope \( \theta \) of the tower modeled as Timoshenko beam can be expressed as

\[
u(x, t) = u_0(x)q(t)
\]

\[
\theta(x, t) = \theta_0(x)q(t)
\]

where \( q(t) = e^{i\omega t} \) is the principal coordinate. The natural frequency is given by \( \omega = \omega_0 + i\omega_1 \) with a real or decaying component \( \omega_0 \) and an imaginary or oscillatory component \( \omega_1 \). For stability, we have \( \omega_0 < 0 \) while \( \omega_1 > 0 \). The spatial terms \( u_0(x) \) and \( \theta_0(x) \) are assumed as

\[
u_0(x) = \sum_{k=1}^{n} a_k \phi_k = \Phi_1 \mathbf{p}_1
\]

\[
\theta_0(x) = \sum_{k=1}^{n} b_k \psi_k = \Psi_2 \mathbf{p}_2
\]

where

\[
\mathbf{p}_1 = [a_1, a_2, \ldots, a_n]^T, \quad \mathbf{p}_2 = [b_1, b_2, \ldots, b_n]^T
\]

\[
\Phi_1 = [\phi_1, \phi_2, \ldots, \phi_n], \quad \Psi_2 = [\psi_1, \psi_2, \ldots, \psi_n]
\]

Here, \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are the matrices containing unknown coefficients \( a_k \) and \( b_k \). \( \Phi_1 \) and \( \Psi_2 \) are the matrices containing trial functions \( \phi_k \) and \( \psi_k \). These trial functions are chosen such that they satisfy the boundary conditions of the tower. In general, they should satisfy the essential boundary conditions. If they also satisfy the natural boundary conditions, more accurate solutions are obtained as shown by Bhat (1985). For the simplicity of the solution, we define three matrices such that

\[
\Phi = \begin{bmatrix} \Phi_1 & 0 \end{bmatrix}
\]

\[
\Psi = \begin{bmatrix} 0 & \Psi_2 \end{bmatrix}
\]

\[
\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\mathbf{p}_2 \end{bmatrix}
\]

Then, in matrix form, displacement and the pure bending slope can be respectively expressed as

\[
u(x, t) = \Phi \mathbf{p} q(t) = \Phi \mathbf{r}
\]

\[
\theta(x, t) = \Psi \mathbf{p} q(t) = \Psi \mathbf{r}
\]

where

\[
\mathbf{r} = \mathbf{p} q(t)
\]

3.1.1. Energy calculation

The energy terms involved in the vibration of ocean tower modeled as Timoshenko beam are described as follows.

3.1.1.1. Potential energy. Potential energy stored in the beam due to bending and shear strain, as explained by Auciello and Ercolano (2004), is given by

\[
U_1 = \frac{1}{2} \int_0^L [EI(\theta')^2 + Gk_o A(u' - \theta')^2] dx
\]

where \( k_o \) is the shear correction factor of the cross section. \( EI \) and \( Gk_o A \) are the flexural and shear rigidity, respectively. Potential energy stored in foundation springs is given by

\[
U_2 = \frac{1}{2} k_o \theta_x^2 |_{x=0} + \frac{1}{2} k_i u_x^2 |_{x=0}
\]

or

\[
U = \frac{1}{2} Tr [K_1 + K_2] \mathbf{r} = \frac{1}{2} Tr K \mathbf{r}
\]

where

\[
K_1 = \int_0^L \left[ \Phi \mathbf{A} \Phi^T + \rho \mathbf{B} \mathbf{C} \mathbf{B}^T \right] dx
\]

\[
K_2 = \int_0^L \left[ \Psi \mathbf{A} \Psi^T + \rho \mathbf{B} \mathbf{C} \mathbf{B}^T \right] dx
\]

3.1.1.2. Kinetic energy. Kinetic energy stored due to translational and rotary inertia of the beam can be given by

\[
T_1 = \frac{1}{2} \int_0^L \rho^* \mathbf{A} \mathbf{u}^T + \rho \mathbf{B} \mathbf{u}^T \theta' dx
\]

where

\[
\rho^* = \rho + C_A \rho_w, \quad 0 \leq x \leq \alpha L
\]

\[
\rho^* = \rho, \quad \alpha L < x \leq L
\]

For underwater part of the beam \( (0 \leq x \leq \alpha L) \), added mass is included using added mass coefficient \( C_A \). Kinetic energy of the tip mass is given by

\[
T_2 = \frac{1}{2} \rho^* \mathbf{B} \mathbf{u}^T + \frac{1}{2} m_p \mathbf{u}_{\mathbf{p}}^T + \frac{1}{2} I_{\mathbf{p}} \theta_{\mathbf{p}}^2 + \frac{1}{2} m_s \mathbf{u}_s^T + \frac{1}{2} I_s \theta_s^2 + \frac{1}{2} m_e \mathbf{e}_e^T \mathbf{e}_e + \frac{1}{2} I_e \theta_e^2
\]

where the subscript ‘\( \mathbf{p}^\prime \)’ implies ‘tip mass’. Total kinetic energy is \( T = T_1 + T_2 \). In matrix form

\[
T = \frac{1}{2} \mathbf{r}^T \left[ \rho^* \mathbf{B} + \rho \mathbf{B}_s \right] \mathbf{r} + \frac{1}{2} \mathbf{r}^T \left[ m_p \mathbf{B}_p + (l_p + m_p \mathbf{e}_p^2) \mathbf{B}_{10} + m_p \mathbf{e}_p \mathbf{B}_{11} + m_e \mathbf{e}_e \mathbf{B}_{11} \right] \mathbf{r}
\]

or

\[
T = \frac{1}{2} \mathbf{r}^T [M_1 + M_2] \mathbf{r} = \frac{1}{2} \mathbf{r}^T M \mathbf{r}
\]

where

\[
M_1 = \int_0^L \mathbf{B}^T \mathbf{A} \mathbf{B} dx, \quad M_2 = \int_0^L [\Psi^T \mathbf{A} \Psi + \rho \mathbf{B} \mathbf{C} \mathbf{B}^T] dx
\]

\[
B_7 = \int_0^L \mathbf{B}^T \mathbf{A} \mathbf{B} dx, \quad B_8 = \int_0^L [\Psi^T \mathbf{A} \Psi + \rho \mathbf{B} \mathbf{C} \mathbf{B}^T] dx
\]

\[
B_9 = \int_0^L [\Psi^T \mathbf{A} \Psi + \rho \mathbf{B} \mathbf{C} \mathbf{B}^T] dx
\]
\[ B_{11} = \left| [\nabla^2 \Phi] \right|_{x=L} \]

It can be seen that the non-linearity in section area \( A \) and 2nd moment of area \( I \) can be handled easily through the use of spatial integration.

3.1.1.3. Rayleigh dissipation factor. Rayleigh dissipation factor, as explained by Goldstein (1980), for foundation damping can be given by

\[ R = \frac{1}{2} c_t \dot{u}_t^2 + \frac{1}{2} c_i \dot{u}_i^2 = 0 \]

Here, \( c_t \) and \( c_i \) are foundation damping constants assumed to be

\[ c_t = \eta_t k_t \quad c_i = \eta_i k_i \]

where \( \eta_t \) and \( \eta_i \) are coefficients of damping constants. In matrix form, \( R \) can be written as

\[ R = \frac{1}{2} \mathbf{t}^T \mathbf{c}_t \mathbf{t} + \frac{1}{2} \mathbf{c}_i \mathbf{t}^T \]

3.1.2. The governing differential equation (GDE)

The functional for free vibration is given by

\[ \Pi = T - U - R \]

Substituting Eqs. (14), (18) and (21) into the functional \( \Pi \) and then using this functional in the Hamilton’s principle one can get the following equation:

\[ \mathbf{M} \ddot{\mathbf{r}} + \mathbf{C} \dot{\mathbf{r}} + \mathbf{K} \mathbf{r} = 0 \]

Putting \( \mathbf{r} = \mathbf{p}(t) \) (from Eq. (10)) leads to the following Quadratic Eigenvalue Problem (QEP)

\[ (\omega^2 \mathbf{M} + \mathbf{C} + \mathbf{K}) \mathbf{p} = 0 \]

The QEP can be solved by a standard linearization approach as explained by Bai et al. (2000). Solving the QEP generates the eigenvalues or natural frequencies explained by Bai et al. (2000). Solving the QEP generates the eigenvectors.

The boundary conditions of the given tower at \( x=0 \) are

\[ M(0) = k_t \dot{u}_t(0) + c_t \dot{u}_t(0) \quad V(0) = -(k_t u(0) + c_t \dot{u}(0)) \]

(31)

Here, \( M \) and \( V \) are bending moment and shear force respectively. At \( x=L \), we have 4 conditions for the continuity of displacement, slope, bending moment and shear force. The boundary conditions at \( x=L \) is given by

\[ M(L) + l_p \ddot{\theta}_p + m_p e_p \dot{u}_p = 0 \]

(33)

where

\[ u_p = u(L) + e_p \dot{\theta}(L) \]

Thus, there are 4 boundary and 4 continuity conditions for the ocean tower.

3.1.4. Trial function

There are various ways of obtaining trial functions. One of them uses uniform beam mode-shape satisfying the same boundary conditions as that of ocean tower (Section 3.1.3).

In Fig. 2(b), a uniform Timoshenko beam has been shown whose one element has been expanded in Fig. 2(a) where the forces and moments acting on it have been shown. Eq. (35) is a force-balance equation while Eq. (36) is a moment-balance equation for the beam element. These two equations form a system of the governing differential equations (GDE) for a uniform Timoshenko beam, given by

\[ \rho^* \Omega \frac{\partial^2 u_F(x,t)}{\partial t^2} - \frac{\partial V_F(x,t)}{\partial x} = Q_f(x,t) \]

(35)

\[ \rho \frac{\partial^2 \dot{\theta}_F(x,t)}{\partial t^2} = \frac{\partial M_f(x,t)}{\partial x} - V_F(x,t) \]

(36)

where the subscript ‘F’ is to denote ‘Trial Function’. For free vibration, external force \( Q_f(x,t) = 0 \). Since the beam is uniform, \( A_F \) and \( I_F \) are uniform.
and $I_L$ are constant. The shear force and the bending moment are expressed as

$$V_t(x, t) = -Gk_0 A_F \theta_{xx}(x, t)$$

(37)

$$M_t(x, t) = El_F \theta_x(x, t)$$

(38)

where

$$\theta_{xx}(x, t) = u_t(x, t) - \theta_t(x, t)$$

(39)

Here, $\theta_{xx}$ is the shear slope of the uniform beam. The GDE for free vibration can be solved by using the method of separation of variables as follows:

$$u_t(x, t) = Y(x)q_t(t)$$

(39)

$$\theta_{x}(x, t) = \Theta(x)q_t(t)$$

(40)

$$q_t(t) = e^{\omega t}$$

(41)

where

$$\omega_F = \omega_{F, R} + i \omega_{F, I}$$

In spatial form, Eqs. (35) and (36) can be reduced to following two equations:

$$\Theta = -\frac{1}{b} \left( Y'' + (a+c)Y' \right)$$

(42)

$$Y'' + dy'' + eY = 0$$

(43)

where

$$d = a + b + c, \quad e = ab$$

Solving the 4th order ODE given in Eq. (43) by the characteristic equation method, we assume $Y(x) = e^{\lambda x}$. For a non-trivial solution, the characteristic equation is

$$\lambda^4 + b\lambda^2 + e = 0$$

(44)

This is a bi-quadratic equation whose solution is given by

$$\lambda = \left\{ \frac{-b \pm \sqrt{b^2 - 4e}}{2} \right\} = \pm \lambda_1, \pm \lambda_2$$

(45)

Thus, there are the four solutions: $\lambda_1$, $-\lambda_1$, $\lambda_2$, and $-\lambda_2$. For the dry part of the tower, the four solutions are denoted by: $\lambda_{1d}$, $-\lambda_{1d}$, $\lambda_{2d}$, and $-\lambda_{2d}$. Then, the general solution of the uniform Timoshenko beam mode-shape can be written as

$$Y(x) = Y_1(x) + c_1 e^{\lambda_1 x} + c_2 e^{-\lambda_1 x} + c_3 e^{\lambda_2 x} + c_4 e^{-\lambda_2 x}$$

(46)

where $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$, $c_7$, $c_8$, $c_9$, and $c_{20}$ are arbitrary constants, which are to be calculated from the boundary conditions. Putting Eq. (46) into Eq. (42), we can obtain pure bending slopes

$$\Theta(x) = \Theta_1(x) = -\frac{1}{b} \left( Y_1''(x) + (a_1 + c)Y_1'(x) \right)$$

(47)

where

$$a_1 = -\frac{\rho A_F \omega_{F, R}^2}{Gk_0 A_F}, \quad a_2 = -\frac{\rho A_F \omega_{F, I}^2}{Gk_0 A_F}$$

The boundary conditions of the uniform Timoshenko beam should be same as that given in Section 3.1.3 for the ocean tower (this is the basic assumption of Rayleigh-Ritz Method for the trial functions). It can be noted that the tip mass ($m_p$) couples the boundary conditions at $x=L$ (Eqs. (33) and (34)). We need to choose the value of $A_F$ and $I_L$ such that these two boundary conditions are satisfied for the same value of $m_p$. Thus, we take

$$A_F = A(L)$$

(48)

$$I_L = I(L)$$

(49)

In spatial form, the boundary conditions at $x=L$ can be written as

$$El_F \Theta_2(L) + \left( I_F + m_p \omega_{F, R}^2 \right) \omega_{F, R} \Theta_2(L) + m_p \rho \omega_{F, I}^2 \psi_5(L) = 0$$

(50)

Now, the same value of $A_F$ and $I_L$ should also satisfy the boundary condition at $x=0$. This can be possible if the spring and damping constants of uniform beam are modified as given below

$$El_F \Theta_1(0) = k_i (k_i + c_i \omega_F) \Theta_1(0)$$

(52)

$$Gk_0 A_F \left( \psi_5(L) - \Theta_2(L) \right) + m_p \rho \omega_{F, I}^2 \Theta_2(L) + m_p \rho \omega_{F, R}^2 \psi_5(L) = 0$$

(51)

This ensures that the uniform beam satisfies the same boundary conditions as that of non-uniform beam (ocean tower). At $x=al$, we have the following continuity conditions:

$$Y_1(2al) = Y_2(2al)$$

(54)

$$Y_1'(2al) = Y_2'(2al)$$

(55)

$$\Theta_1'(2al) = \Theta_2'(2al)$$

(56)

$$Y_1'(2al) - \Theta_1'(2al) = Y_2'(2al) - \Theta_2'(2al)$$

(57)

In matrix form, these boundary conditions can be written as a system of linear equations, given by:

$$[B][C] = 0$$

(58)

where $[B]$ is an $8 \times 8$ matrix and $[C] = [c_{11}, c_{21}, c_{31}, c_{41}, c_{12}, c_{22}, c_{32}, c_{42}]^T$. To solve for the non-trivial solutions Eq. (58), we have

$$[B] = 0$$

(59)

Eq. (59) is the complex frequency equation of the vibrating beam. The real terms and the imaginary terms are separately equated to zero in order to find the decaying and the oscillatory part of the frequency, respectively.

Eq. (59) can be written in the following form

$$f_1(\omega_{F, R}, \omega_{F, I}) + if_2(\omega_{F, R}, \omega_{F, I}) = 0$$

(60)

Eq. (60) can be solved by taking

$$f_1(\omega_{F, R}, \omega_{F, I}) = 0$$

(61)

$$f_2(\omega_{F, R}, \omega_{F, I}) = 0$$

(62)

where $f_1$ and $f_2$ are the non-linear functions of $\omega_{F, R}$ and $\omega_{F, I}$.

The system of non-linear equations in Eq. (61) can be solved to obtain $n$ number of frequencies. Algorithms like ‘trust-region-dogleg’ can be utilized to solve such problems. In MATLAB, the built-in function fsolve uses this algorithm to solve the system of non-linear equations. Now, each frequency obtained can be put in Eq. (58), to form a homogenous system of complex linear equations which can be solved using Single Value Decomposition of matrix $[B]$ to obtain the matrix $[C]$ containing the arbitrary constants. These constants can be put in Eq. (46) and Eq. (47), respectively to obtain uniform beam
mode-shapes $\phi_k$ and uniform pure bending slope mode-shape $\psi_k$. These mode-shapes can be used as trial functions in Rayleigh-Ritz Method as given in Section 3.1.

### 3.2. Finite element method

Considering the Timoshenko beam element of length $L$, displacement and pure bending slope can be respectively assumed as

$$u(\xi, t) = u_0(\xi)q(t)$$

(62)

$$\theta(\xi, t) = \theta_0(\xi)q(t)$$

(63)

$$q(t) = e^{w(t)}$$

(64)

where $u_0(\xi)$ and $\theta_0(\xi)$ are the spatial terms in local coordinate $\xi$ and $q(t)$ is the principal coordinate. Utilizing the Mindlin-type linear beam element of $C^1$-order with four degrees of freedom, as explained in Bathe (1996), displacement and pure bending slope can be written as

$$u(\xi, t) = \left[ u_1 \quad 0 \quad u_2 \quad 0 \right]$$

(65)

$$\theta(\xi, t) = \left[ 0 \quad u_1 \quad 0 \quad u_2 \right]$$

(66)

where

$$\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix} = q(t)$$

where $u_1$ and $u_2$ are nodal displacements, and $\theta_1$ and $\theta_2$ are nodal pure bending slopes. Let $n_e$ denote nth element of the beam and total number of elements be $N_e$. The energy terms for each beam element are described as follows.

#### 3.2.1. Potential energy

Potential energy in bending and shear for an element is given by

$$U^e = \frac{1}{2} \int_0^{L_e} EI(\theta')^2 + GkA(u' - \theta')^2 d\xi$$

(67)

Potential energy stored in foundation springs is given by

$$U^e_2 = \frac{1}{2} k u_2^2 \bigg|_{x=0} + \frac{1}{2} k_2 \theta_2^2 \bigg|_{x=0}$$

(68)

In matrix form

$$U^e = \frac{1}{2} L^T e \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] e$$

(69)

$$U^e_2 = \frac{1}{2} \begin{bmatrix} k_1 \theta_2^2 + k_2 \theta_2^2 \\ k_1 \theta_2^2 + k_2 \theta_2^2 \end{bmatrix}$$

(70)

where

$$\begin{bmatrix} B^e_1 \\ B^e_2 \end{bmatrix} = \int_0^{L_e} A[N]\xi d\xi$$

(71)

Total potential energy in the beam element is

$$U^e = \left\{ \begin{array}{ll} U^e_1 + U^e_2 = \frac{1}{2} [ K^e ] \{ e \} & \text{if } n_e = 1 \\ U^e_1 = \frac{1}{2} [ K^e ] \{ e \} & \text{if } n_e > 1 \end{array} \right.$$
Putting $r = p(t)$ leads to the following Quadratic Eigenvalue Problem (QEP)

$$(\omega^2 M + \omega C + K)p = 0$$

(85)

The QEP can be solved by a standard linearization approach as explained by Bai et al. (2000). This gives the natural frequencies $\omega$ as eigenvalues and mode-shape ($\phi$ and $\psi$) as eigenvectors, where $\phi$ is the displacement mode-shape and $\psi$ is the pure bending slope mode-shape.

### 3.2.4. Axial load

Including the effect of axial load (due to gravity) brings compression due to the self-weight of ocean tower. Thus, the compressive work done by gravity on the beam element is given by as explained by Aucillo and Ercolano (2004)

$$W_{1g}^e = -\frac{1}{2} \int_0^L P(x)(u')^2dz$$

(86)

Where $P(x) = (m_0 + \rho \frac{A}{L} x)g$

Further, work done by gravity due to the eccentricity of the tip mass is given by

$$W_{2g}^e = -\frac{1}{2}m_0ge_{\theta}$$

(87)

In matrix form

$$W_{1g}^e = \frac{1}{2} \int P(x)[ - B_{12}^e]r_e = -\frac{1}{2} \frac{1}{4} K_{12}^e r_e$$

(88)

$$W_{2g}^e = \frac{1}{2} \int m_0 \{ - r_{12}^e \} r_e = -\frac{1}{2} \frac{1}{4} K_{22}^e r_e$$

(89)

Where $B_{12}^e = \int_0^L [N_0^n]^T [N_0^n] A dx$, $B_{11}^e = [N_0^n]^T N_0^n$.

Total work done by gravity on the beam element

$$W_g^e = \begin{cases} W_{1g}^e + W_{2g}^e = \int \frac{1}{2} [K_{12}^e + K_{22}^e] r_e = \frac{1}{2} \frac{1}{4} K_{3}^e r_e \text{ if } n_e = N_e \\ W_{1g}^e = \int \frac{1}{2} [K_{12}^e] r_e = \frac{1}{2} \frac{1}{4} K_{3}^e r_e \text{ if } n_e < N_e \end{cases}$$

(90)

Total work done by gravity on the beam is given by

$$W_e = \sum_{e=1}^{N_e} W_e^e = \frac{1}{2} \int K_3^e r_e$$

(91)

The required functional can be modified as

$$H[r] = T - U - W_e - R$$

(92)

Thus, using Hamilton's principle, the GDE can be worked out to be

$$M \ddot{r} + C \dot{r} + (K + K_e^e) r = 0$$

(93)

---

### 4. Results

#### 4.1. Model verification

In this section, the results obtained by the present theory (Rayleigh Ritz Method or RRM) are compared with those in the existing literature and FEM for the verification of the method and computer program. The dimensions and parameters of the ocean tower are taken to be same as those considered by Wu and Chen (2010). The tower has length $L = 30$ m and has uniform cross section with diameter $D = 0.5$ m. The values of physical constants are given by: density of the beam material $\rho = 7850$ kg/m$^3$, diameter of water $\rho_w = 1000$ kg/m$^3$, Young's modulus $E = 206.8$ GPa, Poisson's ratio $\nu = 0.3$, shear modulus $G = E/(2(1 + \nu)) = 79.54$ GPa and shear correction factor $k_0 = 0.75$ (for circular cross section). The reference parameters are given by: reference tip mass $m_{\text{ref}} = \rho AL = 46240.315$ kg, reference tip rotary inertia $I_{\text{ref}} = \rho AL^3 = 41616283.183$ kg m$^2$, reference translational spring constant $k_{\text{ref}} = EI/L^3 = 23498.312$ N/m and reference rotational spring constant $k_{\text{ref}} = EI/L = 21148480.779$ Nm/rad. For the subdomain ratio $\kappa = 0.8$, tip-mass ratio $\gamma = m_{\text{ref}}/m_{\text{tip}} = 1$, rotary inertia ratio $\zeta = I_{\text{ref}}/I_0 = 1$, eccentricity $e_0 = 0.5$ m and added mass coefficient $C_A = 1$, the lowest five natural frequencies are shown in Table 1 for an elastically supported beam with spring stiffness ratios $k' = k_{\text{ref}}/k_0 = k_0/k_{\text{ref}} = 10^0$ and $10^3$. The number trial functions considered in RRM is 10 and for FEM, number of beam elements considered is 200.

It can be seen that the frequencies obtained by analytical method (Wu and Chen (2010)) and RRM for uniform beam show an excellent agreement in results. Since the uniform beam mode-shapes are used as trial functions in RRM, the weight matrices, and reference translational spring constant $k_{\text{ref}} = EI/L^3 = 23498.312$ N/m and reference rotational spring constant $k_{\text{ref}} = EI/L = 21148480.779$ Nm/rad. This has been taken care of through the use of reduced integration technique (1-point Gaussian quadrature) in the computer program while calculating the stiffness matrix of the beam, as explained by Reddy (2004).

#### 4.2. Convergence study

In this section, we will do a convergence study of the results obtained by Rayleigh Ritz Method and Finite Element Method. We consider the non-uniform ocean tower as shown in Fig. 1. The reference parameters are defined by: reference tip mass $m_{\text{ref}} = \rho \int_0^L A dx$, reference tip rotary inertia $I_{\text{ref}} = 3\rho \int_0^L A x^2 dx$, reference translational spring constant $k_{\text{ref}} = EI/L^3$ and reference rotational spring constant $k_{\text{ref}} = EI/L$ where $D_b$ and $L_b$ are the diameter and second moment of cross section area of the beam at $x = 0$. Here, it is important to note that $D_b$ has been used as characteristic length in reference translational and rotational spring constant as it influences the flexural rigidity of the beam which affects the non-classical boundary condition at the foundation. The values of physical constants are same as in previous section. For convenience, the following dimension and parameters of the tower are considered: length $L = 30$ m, submergence ratio $\alpha = 0.8$, tip-mass ratio $\gamma = 1$, rotary inertia ratio $\zeta = 1$, eccentricity $e_0 = 0.5$ m, base diameter $D_0 = 1$ m, tapered beam fraction $\beta = 0.5$, spring stiffness ratio $k' = k_{\text{ref}}/k_0 = k_0/k_{\text{ref}} = 10^0$ and damping coefficient.

---

### Table 1

<table>
<thead>
<tr>
<th>$\kappa'$</th>
<th>Methods</th>
<th>Natural frequencies (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>10</td>
<td>RRM</td>
<td>0.560</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>0.560</td>
</tr>
<tr>
<td>$10^5$</td>
<td>RRM</td>
<td>0.614</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>0.614</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>RRM</td>
<td>0.614</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>0.614</td>
</tr>
</tbody>
</table>

\( \eta = \eta_0 = 10^{-4} \text{ s} \). The convergence study has been done for tip diameter \( D_p = 0.5 \text{ m}, 0.75 \text{ m}, \) and 1 m.

The convergence parameter for \( j \)th frequency \( \Omega_j \) is given by

\[
\Omega_{j,R} = \left. \frac{\text{Im} (\omega_{j,n})}{\text{Re} (\omega_{j,n})} \right|_{n=0}
\]

\[
\Omega_{j,I} = \left. \frac{\text{Im} (\omega_{j,n})}{\text{Re} (\omega_{j,n})} \right|_{n=0}
\]

where \( \Omega_{j,R} \) and \( \Omega_{j,I} \) are the convergence parameters of the real and imaginary part of the frequency respectively, \( i \) is the number of trial functions (for RRM) or elements (for FEM) considered and \( m \) is the number of trial functions or elements at which the frequency converges. Value of \( m \) should be chosen such that \( \Omega_{j,R} \Omega_{j,I} < 1 \) for \( i > m \) where \( \varepsilon \) is a small number. In the present study, \( m = 25 \) for RRM and \( m = 400 \) for FEM has been considered. At this value of \( m \), the convergence parameter \( \Omega_{j,R} \), \( \Omega_{j,I} < 10^{-3} \) for \( i > m \).

Table 2 shows the convergence study of the frequencies obtained using the Rayleigh-Ritz Method. It can be observed that real and imaginary parts of the frequencies converge at different rates. In all the cases, the imaginary part converges faster than the real part, i.e., fewer trial functions are required for convergence of imaginary part. It can be further observed that as the non-uniformity in the beam decreases, i.e., as \( D_p / D_b \) comes closer to 1, the convergence is faster. This is because for a more uniform beam, higher order trial functions (uniform beam mode-shapes) do not contribute significantly to the lower order non-uniform beam mode-shapes.

Table 3 shows the convergence study of the frequencies obtained using the Finite Element Method. In this case too, it can be observed...
that imaginary part of the frequencies converge faster than the real part. Also, the convergence is faster when the beam is more uniform. However, the number of elements at which the solution converges is much higher than the number of trial functions considered in RRM. Due to this, the matrix size in the eigenvalue problem becomes large in FEM which leads to a higher computation time.

4.3. Parametric study

In this section, we will do a parametric study of the ocean tower, shown in Fig. 1, to find the dependence of the natural frequencies on various configurations of the tower.

Table 4 shows the results of Parametric Study 1. A total of 15 cases are considered corresponding to different foundation parameters. For each case, lowest three damped natural frequencies obtained by RRM and FEM. Further, the percentage difference in the results, obtained by two approaches, is shown in brackets. In this study, the foundation parameters are varied: stiffness ratio \(\kappa = k_2/k_{02} = k_1/k_{01} = 10^{-5}, 10^{-4}, 10^{-3}, 10^0, 10^1\) and damping coefficient \(\eta = \eta_1 = \eta_2 = 10^{-5}, 10^{-4}, 10^{-3}\) s. The tip diameter chosen is \(D_t = 0.5\) m. The number of trial function considered for RRM is \(n = 25\) while number of elements considered for FEM is \(N_e = 400\). All other dimensions and parameters are same as considered in Section 4.2. It can be observed that as the stiffness ratio (\(\kappa\)) increases, the foundation becomes stiffer and hence, the imaginary part of the frequency increases. An interesting result is obtained in the real part of the frequency. For low soil stiffness, its magnitude is low. As soil stiffness increases, its magnitude first increases and then again decreases. This is because at low stiffness, the damping is low as damping constant is assumed to be proportional to spring constant. At higher value of spring constant, the foundation becomes stiffer, the motion of the damper becomes negligible and hence the effect of damping reduces. Hence, the maximum damping occurs at the intermediate value of soil stiffness. Next, we look at the effect of damping coefficient (\(\eta\)) on the natural frequencies of the tower. It can be seen that as damping coefficient increases the real part of the frequency increases, indicating that vibration will damp out at faster rate. However, the damping coefficient has negligible effect on the imaginary part of the frequency. Beside this, it is observed that the results obtained by RRM and FEM show a good agreement. The difference in the
imaginary part of the frequencies, obtained by two approaches, is within 0.2%. However, the difference in real part is slightly larger (but within 2%). This can be attributed to the slow convergence of the real part of the frequencies as seen in Section 4.2. Fig. 3 shows the magnitude of normalized complex mode-shapes (with phase correction) of non-uniform ocean tower for three cases: 2, 8 and 14. It can be observed that case 2 represents soil with very low stiffness ratio, case 8 with intermediate stiffness ratio and case 14 with high stiffness ratio (clamped condition).

Table 5 shows the results of Parametric Study 2. A total of 16 cases are considered. In this study too, the lowest three damped natural frequencies obtained by RRM and FEM are shown for each case. Further, the percentage difference in results obtained by the two approaches is also shown in brackets. Following parameters are varied in this study: tip diameter \(D_p = 0.5 \text{ m}, 0.75 \text{ m}\); rotary inertia ratio \(\gamma = 0, 1\); tip mass ratio \(\zeta = 0, 0.5, 1\) and submergence ratio \(\alpha = 0.2, 0.8\). The number of trial functions considered for RRM is \(n = 25\) while number of elements considered for FEM is \(N_e = 400\). All other dimensions and parameters are same as considered in Section 4.2. It can be observed that as the submergence ratio \(\alpha\) increases, the magnitude of the frequency decreases slightly in general. This is due to the “added mass effect” which adds to the inertia of the beam without increasing the stiffness in submerged part. However, the magnitude of real part of fundamental frequency may increase or decrease with increase in the submergence ratio. This trend is confirmed independently by RRM and FEM. Similar trend is observed with the tip-mass ratio \(\gamma\) and rotary inertia ratio \(\zeta\). The tip mass and its rotary inertia add to the inertia of the beam without changing the stiffness. Hence, the increase in tip-mass ratio and rotary inertia ratio leads to significant decrease in the magnitude of the natural frequency of the tower in general. However, the magnitude of the real part of fundamental frequency may increase as can be seen in case 5 and 7. Increasing the tip diameter \(D_p\) makes the structure stiffer; hence, the natural frequency increases in magnitude for all the cases.

Table 6 shows the results of Parametric Study 3. Here, 4 cases have been considered. The lowest three damped natural frequencies obtained by FEM are shown for each case. In this study, effect of axial force due to gravity \(g = 9.81 \text{ m/s}^2\) is considered and the percentage change in frequencies obtained with and without gravity is shown in brackets. Following parameters are varied: tip diameter \(D_p = 0.5 \text{ m}, 0.75 \text{ m}\) and tip-mass ratio \(\gamma = 0.5, 1\). The number of elements considered for analysis using FEM is \(N_e = 400\). All other dimensions and parameters are same as considered in Section 4.2. It can be observed that in presence of gravity, the magnitude of frequencies decreases in general as it induces compressive axial load on the structure. The only exception is the real part of the fundamental frequency which increases in magnitude in presence of gravity. It can further be observed that percentage change in magnitude of frequency is more for case 1 and 2 as compared to case 3 and 4. This is because the compression developed in a thinner beam is more (case 1 and 2) when same axial force is applied, leading larger change in natural frequency.

5. Discussion and conclusions

A free vibration analysis of an ocean tower is presented. The tower was modeled as a partially submerged, non-uniform Timoshenko beam supported by an eccentric tip mass on one end and non-classical damped foundation on other end. The analysis was done using Hamilton’s variational principle based on Rayleigh-Ritz Method (RRM) and Finite Element Method (FEM). In RRM, the trial function was assumed as uniform beam mode-shapes satisfying the boundary conditions of ocean tower. In FEM, the Mindlin-type linear beam element of \(C^0\)-order with four degrees of freedom was used as shape function. In order to avoid shear locking, which occurs for thin beams, reduced integration technique was incorporated in FEM.

For the verification of code and model, some of the results obtained by RRM and FEM were compared with the one in the existing literatures which showed a good agreement. Through the convergence study (Section 4.2), it was shown that the solution obtained by RRM converges at much fewer number of trial functions than the number of elements in FEM. This makes the matrix size, in eigenvalue problem, much smaller which leads to faster computation. Thus, the computational efficiency of RRM
### Table 5

**Parametric Study 2: natural frequencies (rad/s) for non-uniform ocean tower for various values of $D_p$, $\gamma$ and $\alpha$.**

<table>
<thead>
<tr>
<th>Case</th>
<th>$D_p$ (m)</th>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>Methods</th>
<th>Natural frequencies (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\omega_1, \gamma$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>FEM</td>
<td>$-4.95 \times 10^{-7}$</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.2</td>
<td>RRM</td>
<td>$-4.95 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>FEM</td>
<td>$-1.90 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0.2</td>
<td>FEM</td>
<td>$-1.44 \times 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.5</td>
<td>0.2</td>
<td>FEM</td>
<td>$-1.95 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

### Table 6

**Parametric Study 3: effect of gravity on natural frequencies (rad/s) for non-uniform ocean tower for various value of $D_p$ and $\gamma$.**

<table>
<thead>
<tr>
<th>Case</th>
<th>$D_p$ (m)</th>
<th>$\gamma$</th>
<th>Natural frequencies (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\omega_1, \gamma$</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>Without gravity</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Without gravity</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>Without gravity</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Without gravity</td>
</tr>
<tr>
<td>3</td>
<td>0.75</td>
<td>0.5</td>
<td>Without gravity</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Without gravity</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>Without gravity</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Without gravity</td>
</tr>
</tbody>
</table>
is much higher than that of FEM. Further, it was shown that the convergence is faster for a more uniform beam.

A few parametric studies were done to study the influence of various parameters on the natural frequencies of the beam. For verification, the results were obtained using both RRM and FEM.

- In **Parametric Study 1**, foundation parameters were varied. It was found that as stiffness ratio increases, i.e., as the foundation becomes stiffer, the imaginary part of the frequency increases while the real part of the frequency decreases. Further, it was found that the damping coefficient only affects the real part of the frequency and has negligible effect on imaginary part.

- In **Parametric Study 2**, several parameters were varied. It was observed that increase in submergence ratio, tip mass ratio and rotary inertia ratio decreases the magnitude of natural frequency of vibration in general. This is because they increase the inertia of the beam without changing the stiffness. Further, increase in tip diameter made the structure stiffer and hence it increased the natural frequency of vibration.

- In **Parametric Study 3**, effect of axial force due to gravity was studied. It was found that gravity decreases the natural frequency of vibration in general as it induces compressive axial load in the structure. Further, for a thinner beam, percentage decrease in frequency is more than that of thicker beam.

This study underlines the efficacy of Rayleigh-Ritz Method (RRM) in free vibration analysis of an ocean tower with various structural engineering complications. It also compares the RRM with FEA for a wide range of design parameters.

**References**


